Vector Bundle Constructions

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Following Milnor-Stasheff; Consider a topological space B. A *real vector bundle* ξ is a surjection of topological spaces

$$\pi : E \to B,$$

along with the structure of a real vector space on each fiber $\pi^{-1}(b)$ that satisfies the local triviality condition: for all $b \in B$ there is some neighbourhood U some $n \in \mathbb{N}$ and some homeomorphism

$$h : U \times R^n \to \pi^{-1}(U)$$

We call E the total space and B the base space. Note that here the dimension of the fiber can depend on the point. A bundle map or morphism is a continuous map on the total spaces

$$g : E \to E'$$

such that $E_b \mapsto E_{b'}$ isomorphically for some $b, b' \in B$. This condition allows g to descend to a map on B by defining $\bar{g}(b) \ := \ b'$. A bundle is *trivial* if it is isomorphic to $B \times \mathbb{R}^n$.

Every manifold comes equiped with a canonical vector bubndle called the tangent bundle. Every embedded submanifold comes equiped with two canonnical vector bundles, the tangent bundle and the normal bundle. If a manifold has a trivial tangent bundle it is called *parallelizable*.

A *cross-section* of a vector bundle, or section for short, is a map

$$B \to E$$

that sends b to the fiber above b.

A *Euclidean vector bundle* is a real vector bundle with the additional structure of a map

$$\mu : E \to \mathbb{R}$$

such that the restriction of this map to the fibers is positive definite and quadratic. This μ should be thought of as the norm comming from an inner-product on the fibers. The quadratic condition is exactly that the norm comes from an inner product and allows us to uniquely determine the inner product from the metric.

1 Continuous Functors

Denote the large category of finite dimensional real vector spaces *with only isomorphisms* as $\text{Vect}^{\cong}$. This category is enriched over Top by considering $\text{Hom}_{\text{Vect}^{\cong}}(V, W)$ with the what topology...? Then a *continuous functor*

$$T : \text{Vect}^{\cong} \times \cdots \times \text{Vect}^{\cong} \to \text{Vect}^{\cong}$$

is a functor such that all the maps on Hom sets are continuous.

Let T be a continuous functor as above in k variables and let ξ_i be k vector bundles over respective base spaces $B(\xi_i)$ and total spaces $E(\xi_i)$. Then we can construct a new vector bundle over $B = B(\xi_1) \times \cdots \times B(\xi_k)$ with fibers over $b = (b_1, ..., b_k) \in B$ given by $T\big(E(\xi_1)_{b_1}, ..., E(\xi_k)_{b_k}\big)$, which as a set is just

$$E \ := \ \bigsqcup_{b \in B} T\big(E(\xi_1)_{b_1}, ..., E(\xi_k)_{b_k}\big)$$

which has the obvious projection onto B. What is clear is that this is a surjective set map with vector space fibers, so what remains to do is provide an appropriate topology on this total space such that the map is continuous and show the local triviality of this bundle.

Lemma. *There exists a topology making this π into a vector bundle.*

The essence of the proof is showing that as a set this is locally a product and then enforcing the topology that makes it locally a product topologically. Of course the continuity condition on the functor will imply the continuity of the map. It is harder to show, but still true, that this construction also gives a smooth bundle if the original bundles were smooth. We will denote the new bundle by $T(\xi_1, ..., \xi_k)$.

In this way T defines a new functor

$$\hat{T} : \text{Bundles} \to \text{Bundles}$$

a functor from the category of bundles with bundle maps into itself.

2 Pulling Back Bundles

Consider a vector bundle ξ with projection $\pi : E \to B$. If $f : B' \to B$ is an arbitrary continuous map then we can define a bundle $f^*(\xi)$ over B' as follows

$$\begin{array}{ccc} B' \times_{f,\pi} E & \dashrightarrow & E \\ \Big\downarrow{\scriptstyle \pi'} & \lrcorner & \Big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\ f\ } & B \end{array}$$

The total space is defined as $B' \times_{f,\pi} E \ := \ \big\{(b, e) \in B' \times E : f(b) = \pi(e)\big\}$, the projection π' is just projection off the first variable. The induced horrizontal map

$$\hat{f} : B' \times_{f,\pi} E \to E$$

is also just projecting off the second variable. We topologise this using the subspace topology of the product topology? What remains to do is define the vector space structure on the fibers (we should also check continuity of π' and local triviality but its been done). We define for $(b, e_i) \in (\pi')^{-1}(b), t_i \in \mathbb{R}$

$$t_1(b, e_1) + t_2(b, e_2) \ := \ (b, t_1e_1 + t_2e_2).$$

Using this construction if we have a bundle map $g : E(\xi) \to E'(\xi')$ then we can show that there is a bundle isomorphism $\xi \cong \bar{g}^*(\xi')$. In particular *all bundle maps are just pullbacks.*

3 Example Constructions

We will try to give the explicit description and the description in terms of the above two paradigms. Fix some vector bundles ξ_i given by $\pi_i : E(\xi_i) \to B(\xi_i)$.

3.1 Restricted Bundles

If $B' \subseteq B$ then we can *restrict* the bundle over B to B' by simply setting $E' = \pi^{-1}(B'$ and taking the restriction of the projection $\pi' = \pi|_{E'}$. The fibers are clearly identical to the original fibers (there are just less of them) and so we have a new vector bundle.

If we consider the pullback of the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle \pi} \\ B' & \xhookrightarrow[i]{} & B \end{array}$$

then what we get is

$$B' \times_i E = \{(b, e) \in B' \times E : b = \pi(e)\}$$

which is clearly the bundle described above (with the same map too).

3.2 Products

Given ξ_1, ξ_2 we define $\xi_1 \times \xi_2$ as the bundle

$$\pi_1 \times \pi_2 : E_1 \times E_2 \to B_1 \times B_2.$$

Because the fiber of this map is the product of the two fibers,

$$(\pi_1 \times \pi_2)^{-1}(b_1, b_2) = E_{b_1} \times E_{b_2}$$

there is the simple component wise vector space structure. It is clear that this construction corresponds to the continuous functor assigning to two vector spaces their product.

3.3 Whitney Sums

If ξ_i are over the same base then we can pullback along the diagonal map

$$d : B \to B \times B$$

the product of the two bundles $\xi_1 \times \xi_2$

$$\begin{array}{ccc} & & E_1 \times E_2 \\ & & \downarrow{\scriptstyle \pi} \\ B & \xhookrightarrow[d]{} & B \times B \end{array}$$

this is called the Whitney sum and denoted $\xi_1 \oplus \xi_2$. Note that pulling back along the diagonal is the same as pulling back along the *inclusion* of the diagonal as a subspace and so we see that this is also a restriction of the product bundle. The fibers are canonically isomorphic to the direct sum of fibers and so this is also the bundle comming from the application of the continuous direct sum functor on vector spaces, and then restricting to the diagonal. This is also clear since the direct sum is a biproduct for vector spaces and since this is a restriction of the product bundle it must have the same fibers (that are products or direct sums).

If we now consider sub-vector bundles, that is bundles such that $E(\xi_1) \subseteq E(\xi_2)$ and the fibers of ξ_1 are vector subspaces of those of ξ_2 then we can ask when a bundle can be written as the direct (Whitney) sum of two sub-bundles.

Lemma. *If $\xi_1, \xi_2 \subseteq \eta$ are sub-bundles such that for every fiber we have*

$$E(\eta)_b = E(\xi_1)_b \oplus E(\xi_2)_b$$

then $\eta \cong \xi_1 \oplus \xi_2$, that is if a bundles fibers are the direct sum of sub-bundles then the bundle is the direct sum of those sub-bundles.

If η is a Euclidean bundle and $\xi \subseteq \eta$ is a sub-bundle then we can define the *orthogonal bundle* of ξ fiber wise by specifying that

$$E(\xi^\perp)_b = E(\xi)_b^\perp$$

We then define the total space $E(\xi^\perp)$ as the union of these fibers. Note that to take the orthogonal compliment of $E(\xi)_b$ we are using the inner-product structure. We give this total space the subspace topology from the total space of *eta*.

Lemma. *$E(\xi^\perp)$ is the total space of a sub-bundle $\xi^\perp \subseteq \eta$ moreover*

$$\eta \cong \xi \oplus \xi^\perp$$

Remark: Smooth manifolds can always be given a Riemannian metric (that is a euclidean metric on the tangent bundle) and so the tangent bundle can always be written as direct sums given some sub-bundle. One way of getting a sub-bundle to the tangent bundle is to consider the tangent bundle of a sub-manifold, in this case the orthogonal compliment is called the *normal bundle*.

3.4 Hom

We look here only at the dual bundle. If we have a base space B and we consider the continuous functor given by *dualising*

$$V \mapsto \mathrm{Hom(V, \mathbb{R})}$$

then this induces the functor on the category of bundles over B given by

$$\xi \mapsto \mathrm{Hom}(\xi, \epsilon^1)$$

where ϵ^1 denote the trivial line bundle over B.